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problem was first proposed and solved in 1803 by Malfatti, and is generally known by his name. A very beautiful construction was given by Steiner in 1826 with no proof, but the appended remark that this construction showed how powerful were the methods which the writer had developed.¹ Steiner's construction served as a challenge to geometers, and many proofs were attempted. By far the simplest and most elegant was given in 1856 by Hart.² The German mathematicians were ill pleased that the first good proof should be in English; perhaps they failed to appreciate the strategic significance of the fact that Hart was an Irishman. At any rate they objected that Hart had used methods unknown to Steiner.³ This criticism seems to us trivial. Steiner was a sly old fox, who probably knew a good deal more mathematics than he ever put on paper. For instance, there seems good reason to believe that he was familiar with inversion in a circle, though he did not give it as one of his working methods.⁴ As for simplicity, one has but to compare Hart's proof as given, let us say, in *Casey's Sequel to Euclid*,⁵ with the proof in Petersen's little classic on geometrical constructions⁶ to see how immeasurably superior the former is. Analytic determinations of the radii and points of contact of the circles have not been wanting; that given by Professor Gummer in the MONTHLY, Vol. XXIV, No. 3, being quite as simple as any. On the other hand the Lemoine geometrographic numbers called for by Steiner's construction can be reduced to the remarkably small proportions of Simplicity 66, Exactitude 42.⁷

502. Proposed by R. P. BAKER, University of Iowa.

A designer of machinery requires a curve having the following properties:

- (1) A closed curve touching a given circle at two diametral points and enclosing it.
- (2) The sum of the three radii from the center of this circle to the curve which make with each other angles of 120° is constant.
- (3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

SOLUTION BY TOBIAS DANTZIG, Indiana University.

Let (Fig. 1) AOA' be the diameter of contact and BOB' a perpendicular diameter. I shall seek a continuous algebraic curve symmetric with respect to the two diameters satisfying the conditions of the problem. If $\rho = f(\theta)$ is the equation of the curve, f is a function with a period π in θ and consequently of the form

$$(1) \quad \rho = a + c \sin^2 \theta + e \sin^4 \theta + \dots,$$

where a is evidently the radius of the circle. If we stop at the first term, we obtain the trivial solution of the circle itself, which evidently satisfies all the conditions of the problem. If we take two terms,

$$(2) \quad \rho = a + c \sin^2 \theta,$$

we obtain a circular sextic whose cartesian equation is

$$(3) \quad (x^2 + y^2)^3 = (ax^2 + by^2)^2,$$

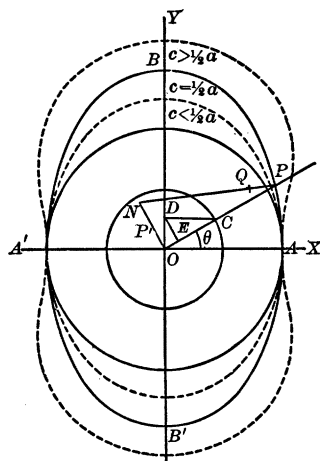


FIG. 1.

¹ "Einige geometrische Betrachtungen," *Crelle*, Vol. 1, 1826.

² "Geometrical Investigation of Steiner's Solution of Malfatti's Problem," *Quarterly Journal of Mathematics*, Vol. 1, 1856.

³ SCHROETER, *Crelle*, Vol. LXXVII, 1874, p. 232.

⁴ BÜTZBERGER, *Ueber bizentrische Polygone*, Leipzig, 1913, pp. 50ff.

⁵ P. 149 in the first ed., Dublin, 1881.

⁶ Fourth edition of the French translation, *Méthodes et théories pour la résolution des problèmes de constructions géométriques*, Paris, 1908, pp. 103ff.

⁷ HAGGE, "Zur Konstruktion der Malfattischen Kreise," *Zeitschrift für mathematische Unterricht*, Vol. XXXIX, 1908, p. 588.

where $b = a + c$. We have here

$$x^2 = \frac{\rho^2(b - \rho)}{c}, \quad y^2 = \frac{\rho^2(\rho - a)}{c}$$

and the maximum value for x occurs for $\rho = \frac{2}{3}b$. According as $c \geq a/2$, this maximum value really occurs, is coincident with A , or is imaginary. The three cases are illustrated in Fig. 1. That condition (2) is satisfied for the curve in question is easy to verify. Indeed, let ρ_1, ρ_2, ρ_3 be three radii inclined 120° to each other. Then

$$\rho_1 + \rho_2 + \rho_3 = 3a + c \left[\sin^2 \theta + \sin^2 \left(\theta + \frac{2\pi}{3} \right) + \sin^2 \left(\theta - \frac{2\pi}{3} \right) \right] = 3a + \frac{3}{2}c = \frac{3(2a + c)}{2}.$$

The angle φ which the normal makes with the radius vector has for tangent, ρ'/ρ , and $\rho' = 2c \sin \theta \cos \theta$. The curve can be constructed by points in the following manner. Let c be the point where the circle of radius c meets an arbitrary radius. Project C on OY in D , and back on the radius in E . Then $OE = c \sin^2 \theta$ and if we carry $EP = a$, the point P is on the curve. If we now draw at O a perpendicular to OP and carry $ON = 2DE = 2c \sin \theta \cos \theta$, then NP is the normal at P .

The linkage of Fig. 2 enables one to construct the locus of P as well as that of any point Q on its normal by a continuous motion. O, F , and G are fixed pivots. $OF = a$. $FG = GH = GH' = \frac{1}{2}c$. Through H and H' slides the bar HPH' and GH' is kept parallel to OY . The point P , intersection of OG with HH' , describes the curve.

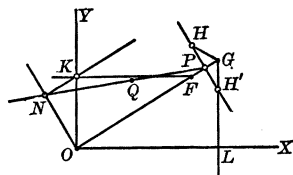


FIG. 2.

In the second part of the linkage, FK is kept parallel to OX , ON is rigidly fixed at right angles to OG and K is a loose pivot, while N slides on ON . If, in the motion of the radius OG , K is describing OY , the bar NP is enveloping the evolute of the curve P and any point Q at a constant distance $PQ = d$ will describe the locus sought in the problem.

503. Proposed by J. W. CLAWSON, Ursinus College, Penn.

If two points A and B invert with respect to a third point O as center of inversion into A' and B' , the middle point of the segment AB inverts into the point other than O where the circle of Apollonius (the locus of a point P moving so that $A'P/PB = A'O/OB'$) cuts the circle $OA'B$.

SOLUTION BY HORACE OLSON, Chicago, Illinois.

By elementary geometry it is evident that the line AB inverts into the circle passing through O, A' , and B' . Let C be the middle point of AB , and C' the point into which it inverts. Draw the lines $A'C'$ and $C'B'$. Triangle ACO is similar to triangle $C'A'O$, and triangle BCO to triangle $C'B'O$. Hence,

$$\frac{A'C'}{AC} = \frac{A'O}{CO}, \quad \text{and} \quad \frac{C'B}{CB} = \frac{B'O}{CO};$$

whence (since $AC = CB$)

$$\frac{A'C'}{C'B'} = \frac{A'O}{B'O},$$

and the proposition is proved.

Also solved by FRANK IRWIN and R. A. JOHNSON.

CALCULUS.

420. Proposed by W. J. GREENSTREET, Stroud, England.

The join of the center of curvature of a curve to the origin is at an angle α to the initial line. Prove that with the usual notation,

$$\frac{d\alpha}{d\psi} \left[\left(\frac{dp}{d\psi} \right)^2 + \left(\frac{d^2p}{d\psi^2} \right)^2 \right] = \frac{dp}{d\psi} \cdot \frac{d\rho}{d\psi}.$$

